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Continuum Argumentation Frameworks from Cooperative Game Theory

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Abstract. We investigate the argumentation frameworks (AFs) that arise from multi-player transferable-utility cooperative games. These AFs have uncountably infinitely many arguments; arguments represent alternative payoff distributions to the players. We examine which of the various properties of AFs (from Dung's 1995 seminal paper) hold; we prove that these AFs are never finitary, never well-founded, always controversial and never limited controversial. We hope that this will encourage further exchange of ideas between argumentation and cooperative games.

Keywords. Abstract argumentation, cooperative game theory, dialogue

1. Introduction

Abstract argumentation theory is the branch of artificial intelligence (AI) concerned with resolving conflicts between disparate claims in a transparent and rational manner, while abstracting away from the contents of such claims by focussing on how they disagree (e.g. [6,12]). The resulting directed graph (digraph) representation of arguments (nodes) and the attacks between them (directed edges), called an *abstract argumentation framework* (AF), resolves conflicts by selecting suitable subsets of arguments, called *extensions*; this has been used to further understand and unify many areas within and outside of AI (see, e.g. [6,12]), where a situation can be represented by some AF such that the resulting extensions correspond to solutions for that situation; this gives a dialectical perspective to the situation that has been applicable to many practical domains (e.g. [11]).

Moreover, the "correctness" of argumentation theory has been shown by demonstrating that a correspondence exists between abstract argumentation theory and *cooperative game theory* (e.g. [5]), the branch of game theory (e.g. [20]) where agents that interact strategically can also work together under binding contracts [5, page 7] to earn more payoff than they can otherwise. This correspondence was first articulated in [6, Section 3.1], and then developed in [22], which further reinforces the applicability of concepts in abstract argumentation to problems of societal concern, but also allows for a cross-fertilisation of concepts between argumentation and cooperative games.

Abstract argumentation theory has mostly considered AFs that have a finite number of arguments (e.g. [1]). AFs that have an infinite number of arguments have not been considered as often, but they have been implicitly investigated in that all the results in [6] also hold for infinite AFs. Properties of sets of winning arguments in infinite AFs

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were further investigated in [3], albeit in an abstract setting. It has been shown in [22] that uncountably infinite (continuum) AFs arise naturally from cooperative games. In this paper, we study these continuum AFs in their own right by asking whether various properties defined in [6, Section 2] hold for these AFs. We prove that these AFs fail to possess several desiderata due to the density of the continuum, specifically *finitariness* (all arguments have finitely many attackers), *well-foundedness* (there are no infinitely long "backwards" chains of attackers), *non-controversy* (no argument can simultaneously (indirectly) attack and defend other arguments), and *limited controversy* (there are no infinite chains of controversies); this makes precise the claim that these AFs are non-trivial, because one cannot invoke these properties to reduce the multiplicity of solutions.

The paper is structured as follows. In Section 2 we recap the relevant aspects of cooperative game theory and abstract argumentation theory. In Section 3, we investigate the properties of the continuum AFs arising from cooperative games and present our main results. We conclude with related and future work in Section 4.

2. Background

Notation: If X is a set, its *power set* is $\mathscr{P}(X)$ and its *cardinality* is $|X|$. $\mathbb{N}$ ($\mathbb{N}^+$) is the *set of (resp. positive) natural numbers*, with $|\mathbb{N}| =: \aleph_0$. $\mathbb{R}$ ($\mathbb{R}_0^+$ / $\mathbb{R}^+$) denotes the set of all (resp. non-negative / positive) real numbers, all with cardinality $2^{\aleph_0}$. For $a,b \in \mathbb{R}$, *the open interval from* a *to* b is the set $(a,b) := \{x \in \mathbb{R} | a < x < b\}$. For $n \in \mathbb{N}$, the *n-fold Cartesian power of* X is X^n, e.g. $X^2 = X \times X$. For sets Y and Z, and functions $f : X \to Y$ and $g : Y \to Z$, $g \circ f : X \to Z$ is *the composition of* f *then* g. $X \hookrightarrow Y$ denotes there is an injection from X to Y, including the case $X \subseteq Y$. For a function $f : X \to \mathbb{R}$, $f \geq 0$ abbreviates $(\forall x \in X)\, f(x) \geq 0$. An X*-sequence* is a function $\mathbb{N} \to X$, denoted as $\{x_k\}_{k\in\mathbb{N}}$.

2.1. Cooperative Game Theory

We review the basics of cooperative game theory (see, e.g. [5,22]). Given $m \in \mathbb{N}^+$, the **set of players** or **agents** is $N := \{1,2,3,\ldots,m\}$. Clearly, $|N| = m$. A **coalition** is any set $C \subseteq N$, where the **empty coalition** is $\varnothing$ and the **grand coalition** is N itself; each such C denotes that the players in C are cooperating under some contract. The **valuation function** $v : \mathscr{P}(N) \to \mathbb{R}$ such that $v(\varnothing) = 0$; $v(C)$ is C's payoff (in arbitrary units) as a result of the agents in C coordinating their strategies as agreed; this measures how "good" each C is. A **(cooperative) (m-player) game (in normal form)** is the pair $G := \langle N, v\rangle$.

The following properties are standard in the literature for v. We say v is **non-negative** iff $v \geq 0$. We say v is **monotonic** iff $(\forall C, C' \subseteq N)\,[C \subseteq C' \Rightarrow v(C) \leq v(C')]$. We say v is **constant-sum** iff $(\forall C \subseteq N)\, v(C) + v(N - C) = v(N)$. We say v is **super-additive** iff $(\forall C, C' \subseteq N)\,[C \cap C' = \varnothing \Rightarrow v(C \cup C') \geq v(C) + v(C')]$. We say v is **inessential** iff $\sum_{k=1}^{m} v(\{k\}) = v(N)$. For the rest of the games in this paper, we will assume that v is non-negative, super-additive and essential (i.e. not inessential). Intuitively, this means there is an incentive to cooperate such that agents working together will earn strictly more (as a coalition) than when working separately.

Given v, what coalitions will form? A **coalition structure**, CS, is a partition of N. As each coalition C earns a payoff $v(C) \geq 0$, we are interested in asking which ways of dividing $v(C)$ amongst the players $k \in C$ are "sensible". In this paper, we consider **transferable utility (TU) games**, which allows for *any* distribution of $v(C)$ to the players in C.[2] An **outcome** of a game is a pair $(CS, \mathbf{x})$, where CS is a coalition structure and $\mathbf{x} \in \mathbb{R}^m$ is a **payoff vector** that distributes the value of each $C \in CS$ to each $k \in C$. As usual in cooperative games (e.g. [5]), we focus on the case where $CS = \{N\}$, i.e. where all agents work together to form the grand coalition, and consider how the resulting payoff $v(N)$ can be distributed to each of the m players via the vector $\mathbf{x}$.

How should $v(N)$ be distributed amongst the m players? We say the payoff vector $\mathbf{x} := (x_1, \ldots, x_m) \in \mathbb{R}^m$ is **feasible** iff $\sum_{k \in N} x_k \leq v(N)$, **efficient** iff $\sum_{k \in N} x_k = v(N)$, **individually rational** iff $(\forall k \in N)\, v(\{k\}) \leq x_k$ and an **imputation** iff $\mathbf{x}$ is efficient and individually rational. We denote the **set of imputations** for a game G with $IMP(G)$, or just IMP if G is clear from context [6]. If G is inessential, then $IMP(G)$ is a singleton set by individual rationality, consisting of just $(v(\{1\}), v(\{2\}) \ldots, v(\{m\}))$, Otherwise, $IMP(G)$ is uncountably infinite; we focus on essential games to avoid this trivialisation.[3]

The solution concepts of cooperative games that we will consider are concerned with whether coalitions of agents are incentivised to defect from the grand coalition because they can earn strictly more payoff. Given a game $G = \langle N, v\rangle$, let $C \subseteq N$ and $\mathbf{x}, \mathbf{y} \in IMP$. We say **x dominates y via** C, denoted $\mathbf{x} \to_C \mathbf{y}$, iff (1) $(\forall k \in C)\, x_k > y_k$ and (2) $\sum_{k \in C} x_k \leq v(C)$, i.e. the agents are (1) strictly better off in C because (2) they will be earning enough as a coalition to be able to split the earnings among themselves. We call C the **defecting coalition**. It is easy to see that for any C, the binary relation $\to_C$ on IMP is irreflexive, acyclic, antisymmetric and transitive. Further, it can be shown that $\to_N = \varnothing$, $(\forall k \in N) \to_{\{k\}} = \varnothing$ and $\to_\varnothing = IMP^2$ (the total relation on IMP). It follows that if $m < 3$, $\to_C = \varnothing$ for any coalition C. The relation $\to$ is irreflexive, but not in general complete, transitive or acyclic (e.g. [19, Chapter 4]). Each cooperative game thus gives rise to an associated digraph, $\langle IMP, \to\rangle$, called an **abstract game**. The domination relation is empty for $m < 3$, so we will consider $m \geq 3$ to avoid this trivialisation.

We now review the solution concepts of cooperative games that are relevant to this paper.[4] Let $I \subseteq IMP$. Define the **forward set of** I to be $I^+ := \{\mathbf{y} \in IMP |\, (\exists \mathbf{x} \in I)\, \mathbf{x} \to \mathbf{y}\}$. If $I = \{\mathbf{x}\}$, then we write $\mathbf{x}^+ := \{\mathbf{x}\}^+$. Dually, we define the **backward set of** I, $I^- := \{\mathbf{y} \in IMP |\, (\exists \mathbf{x} \in I)\, \mathbf{y} \to \mathbf{x}\}$, and $\mathbf{x}^-$ is when $I = \{\mathbf{x}\}$. Define a function $U : \mathscr{P}(IMP) \to \mathscr{P}(IMP)$ to be $U(I) = IMP - I^+$. We say I is a **(von-Neumann-Morgenstern) stable set** iff $I = U(I)$ [20]. We say I is a **subsolution** iff $I \subseteq U(I)$ and $I = U^2(I) := U \circ U(I)$ [14]. We say I is the **supercore** iff I is the $\subseteq$-least subsolution [14]. We say I is the **core** iff $I = \{\mathbf{x} \in IMP | \mathbf{x}^- = \varnothing\}$, i.e. the set of all undominated imputations [7]. Lucas has shown that stable sets may not exist for cooperative games [9,10], although subsolutions, the supercore and the core always exist [13,14,15], but the core can be empty [4,18] exactly when the supercore is empty [14,21,22]. Each of these solution concepts offer alternative "socially acceptable" ways of distributing payoff to the players [20]. We now give two examples to illustrate some of these concepts.

Example 2.1. *[6, page 336] Let $N = \{1, 2, 3\}$ and $v(C) = 0$ if $|C| \leq 1$, and $v(C) = 2$ if $|C| \geq 2$. We show that $I = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is a stable set. Showing that $I \subseteq U(I)$ is equivalent to showing that no two elements in I dominate each other, which is true because we cannot have two components of $\mathbf{x} \in I$ being strictly greater than the two*

[2]The formalism of N and $v : \mathscr{P}(N) \to \mathbb{R}$ is different for non-TU games, see (e.g.) [5, Chapter 5].

[3]This corrects a minor error in [22, Corollary 1], where the assumption of $m \geq 2$ was omitted.

[4]These are *defection-based* solution concepts, whereas solution concepts based on *marginal contributions* (e.g. [17]) are currently outside the scope of this work.

corresponding components of $\mathbf{y} \in I$*, and considering two components suffices because* $\to_C = \varnothing$ *if* $|C| = 1$ *or* $C = N$*. To show that* $U(I) \subseteq I$*, it is equivalent to showing that every imputation* $\mathbf{x} = (x_1, x_2, x_3) \in IMP - I$ *is attacked by some imputation in I. By definition, we have* $0 \leq x_k \leq 2$ *for all* $k \in N$ *and* $x_1 + x_2 + x_3 = 2$*. Either (1)* $x_3 = 0$*, (2)* $x_3 > 1$ *or (3)* $x_3 \in (0,1)$*. (1) implies that* $x_1 + x_2 = 2$*, but as* $\mathbf{x} \notin I$*, WLOG assume* $x_1 < 1$ *and* $x_2 > 1$*, then* $(1,0,1) \to_{\{1,3\}} \mathbf{x}$*. Similarly, if* $x_1 > 1$ *and* $x_2 < 1$*, then* $(0,1,1) \to_{\{2,3\}} \mathbf{x}$*. (2) means* $x_1 + x_2 < 1$ *hence* $(1,1,0) \to_{\{1,2\}} \mathbf{x}$*. (3) means* $x_1 + x_2 < 2$*. If* $x_1 \geq 1$*, then* $x_2 < 1$ *hence* $(0,1,1) \to_{\{2,3\}} \mathbf{x}$*. If* $x_1 < 1$ *then* $x_2 \geq 1$ *so* $(1,0,1) \to_{\{1,3\}} \mathbf{x}$*. In all cases, some imputation in I dominates* $\mathbf{x}$*. Therefore, I is a stable set.*

Example 2.2. *[14, Example 5.1] Consider* $N = \{1,2,3\}$ *with* $v(\{1,2\}) = v(\{3,1\}) = v(N) = 1$*, and for all other S,* $v(S) = 0$*. We claim that* $I := \{(1,0,0)\}$ *is the core and that it is disconnected w.r.t.* $\to$*. Suppose* $(x_1, x_2, x_3) \to_C (1,0,0)$ *for some* $C \subseteq N$*, which is only possible for* $|C| = 2$*. As* $x_1 + x_2 + x_3 = 1$ *and* $x_k \geq 0$ *for* $k = 1,2,3$*, we cannot have* $x_1 > 1$ *and hence* $1 \notin C$*, so the only possible coalition is* $C = \{2,3\}$*, but then* $v(\{2,3\}) = 0$*, hence* $0 \leq x_2 + x_3 \leq 0$*, which means* $x_2 = x_3 = 0$*; this violates the domination condition* $x_2, x_3 > 0$*, hence for all* $\mathbf{x} \in IMP$*,* $\mathbf{x} \not\to (1,0,0)$*, therefore* $(1,0,0)$ *is amongst the undominated imputations. To show that* $(1,0,0)$ *is the only undominated imputation, consider* $(x_1, x_2, x_3) \notin I$*, hence for some* $\varepsilon > 0$*,* $x_1 = 1 - \varepsilon$ *and* $x_2 + x_3 = \varepsilon$*. Either (1) one of* x_2*,* x_3 *is zero or (2) neither are zero. In case (1), WLOG say* $x_2 = 0$*, then* $(x_1, x_2, x_3) = (1-\varepsilon, 0, \varepsilon)$ *which is dominated by* $\left(1 - \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, 0\right)$ *with defecting coalition* $\{1,2\}$*. In case (2),* $(x_1, x_2, x_3) = (1-\varepsilon, x_2, \varepsilon - x_2)$ *for* $x_2 > 0$*, which is dominated by* $\left(1 - \frac{2\varepsilon}{3}, x_2 + \frac{\varepsilon}{3}, \frac{\varepsilon}{3} - x_2\right)$ *with defecting coalition* $\{1,2\}$*. Therefore, I is the core. Now suppose* $(1,0,0) \to_C (x_1, x_2, x_3)$*, but we know that* $x_2, x_3 \geq 0$ *so we cannot have* $0 > x_2, x_3$*, therefore* $2,3 \notin C$*, hence* $C = \{1\}$ *is the only possibility, but* $\to_{\{k\}} = \varnothing$ *for all* $k \in \mathbb{N}$*. Therefore,* $(1,0,0) \not\to \mathbf{x}$ *for all* $\mathbf{x} \in IMP$*. Hence* $(1,0,0)$ *is disconnected from all other imputations w.r.t.* $\to$*.*

2.2. Abstract Argumentation Theory

Recall that an **(abstract) argumentation framework** (AF) is a digraph $\langle A, R \rangle$, where A is **the set of arguments** and $R \subseteq A^2$ is **the attack relation** [6], where $(a,b) \in R$, alternatively denoted as $R(a,b)$, means argument a disagrees with argument b. Let $S \subseteq A$ for the remainder of this subsection. Define the **forward set of** S to be $S^+ := \{b \in A \,|\, (\exists a \in S) R(a,b)\}$. The **neutrality function** $n : \mathscr{P}(A) \to \mathscr{P}(A)$ is defined as $n(S) := A - S^+$. We say S is a **stable extension** iff $S = n(S)$. We say S is a **complete extension** iff $S \subseteq n(S)$ and $S = n^2(S) := n \circ n(S)$. We say S is a **preferred extension** iff it is a $\subseteq$-maximal complete extension. We say S is the **grounded extension** iff it is the $\subseteq$-least complete extension. We say S is the **set of all unattacked arguments** iff $S = \{a \in A | a^- = \varnothing\}$, where $S^- = \{a \in A \,|\, (\exists b \in S) R(b,a)\}$ and $a^- := \{a\}^-$. Stable extensions may not exist for AFs, although complete extensions always exist. Grounded, complete, preferred and stable extensions are collectively called the **Dung semantics**, and each defines a way of resolving the conflicts represented by R.

We say an AF $\langle A, R \rangle$ is **finitary** iff $(\forall a \in A) \, |a^-| < \aleph_0$. An AF is **well-founded** iff there is no A-sequence $\{a_k\}_{k \in \mathbb{N}}$ such that $(\forall k \in \mathbb{N}) R(a_{k+1}, a_k)$; if an AF is well-founded, then its grounded extension is stable [6, Theorem 30] and therefore there is only one subset of winning arguments. For $a, b \in A$, we say a is **indirectly attacking (defending)** b iff there is an odd (respectively, even)-length path from a to b. We say a is

controversial with respect to b iff a both indirectly attacks and indirectly defends b. We say a is controversial iff $(\exists b \in A)\, a$ is controversial w.r.t. b. An AF is **controversial** iff it has a controversial argument, else it is **uncontroversial**. An AF is **limited controversial** iff there is no A-sequence $\{a_k\}_{k\in\mathbb{N}}$ such that $(\forall k \in \mathbb{N})\, a_{k+1}$ is controversial w.r.t. a_k.

By interpreting $\langle IMP, \rightarrow\rangle$ as an AF, it has been shown that Dung's abstract argumentation semantics correspond to the solution concepts of cooperative games:

Abstract Argumentation	Cooperative Game	Reference
Argumentation Framework $\langle A, R\rangle$	Abstract Game $\langle IMP, \rightarrow\rangle$	[6, Section 3.1]
All unattacked arguments	The Core	[6, Theorem 38]
The Grounded Extension	The Supercore	[22, Theorem 5]
Complete Extensions	Subsolutions	[22, Theorem 3]
Preferred Extensions	$\subseteq$-maximal Subsolutions	[6, Section 3], [22, Theorem 3]
Stable Extensions	Stable Sets	[6, Theorem 37]

Table 2.1. Summarising the Correspondence Between Abstract Argumentation and Cooperative Game Theory

3. Some Properties of these Continuum Argumentation Frameworks

Having recapped how $\langle IMP, \rightarrow\rangle$ can be interpreted as an AF with uncountably infinitely many arguments, we now study these AFs in their own right, specifically whether these AFs satisfy or fail to satisfy the various properties defined by Dung in [6, Section 2], which we have recapped in Section 2.2. We prove that these AFs are not finitary, not well-founded, not limited controversial and not uncontroversial. This is due to the continuum nature of IMP arising from transferable utility, and shows that these AFs are not trivial in that we cannot appeal to these properties to conclude other properties that may reduce the multiplicity of the sets of winning arguments [6, Section 2].

Before we begin, let us recapitulate a simplification that does not lose generality. Let $\langle N, v\rangle$ be a game with abstract game $\langle IMP, \rightarrow\rangle$. We can convert it to its $(0,1)$**-normalised form**, which is the game $\left\langle N, v^{(0,1)}\right\rangle$, via the following affine transformation: $v^{(0,1)}(C) := Kv(C) + \sum_{k\in C} c_k$, where $\frac{1}{K} := v(N) - \sum_{k\in N} v(\{k\})$ and $(\forall k \in N)\, c_k := -Kv(\{k\})$. It follows that $(\forall k \in N)\, v^{(0,1)}(\{k\}) = 0$ and $v^{(0,1)}(N) = 1$. Further, the abstract game arising from the $(0,1)$-normalised form is digraph-isomorphic to $\langle IMP, \rightarrow\rangle$, and hence the solution concepts mentioned in Section 2.1 are preserved [2, Definition 2.7]. WLOG, we may assume that IMP is the **standard** $(m-1)$**-dimensional simplex**, $\{(x_1, \ldots, x_m) \in \mathbb{R}^m | (\forall 1 \le k \le m) x_k \ge 0, \sum_{k=1}^m x_k = 1\}$. Further, we will invoke the **Cantor-Schröder-Bernstein (CSB) theorem** (see, e.g. [8, Theorem 3.2]), which states that for (not necessarily finite) sets A and B, if $A \hookrightarrow B \hookrightarrow A$, then A and B have the same cardinality, in which case we write $A \cong B$. We assume standard results from set theory such as $(0,1) \cong \mathbb{R} \cong \mathbb{R}^m$ for every $m \in \mathbb{N}^+$.

First recall that the simplex is closed under affine combinations of two imputations $\mathbf{x}$ and $\mathbf{y}$, as imputations are vectors in $\mathbb{R}^m$ that can be added and scaled. Further, the imputations strictly in between $\mathbf{x}$ and $\mathbf{y}$ can be parameterised uniquely by $(0,1)$.

Lemma 3.1. *Let $t \in (0,1)$ and $\mathbf{x}, \mathbf{y} \in IMP$ be distinct. We have that $(1-t)\mathbf{x} + t\mathbf{y} \in IMP$ and $(0,1) \hookrightarrow IMP$ with rule $t \mapsto (1-t)\mathbf{x} + t\mathbf{y}$ is a well-defined injection.*

Proof. $t \in (0,1)$ implies $t, (1-t) > 0$. (Individual rationality) As each component is of the form $(1-t)x_k + ty_k$, we have $(1-t)x_k + ty_k \ge 0$ because $x_k, y_k \ge 0$, for all $k = 1, \ldots, m$. (Efficiency) $\sum_{k=1}^m [(1-t)x_k + ty_k] = (1-t)\sum_{k=1}^m x_k + t\sum_{k=1}^m y_k = 1 - t + t = 1$.

Assume for contradiction that $t \mapsto (1-t)\mathbf{x}+t\mathbf{y}$ is not injective. Therefore, there exists $t,t' \in (0,1)$ distinct such that $(1-t)\mathbf{x}+t\mathbf{y} = (1-t')\mathbf{x}+t'\mathbf{y}$. Basic algebra means we have $(t'-t)\mathbf{x} = (t'-t)\mathbf{y}$, but as $t'-t \neq 0$, it follows $\mathbf{x} = \mathbf{y}$, which is a contradiction. □

Clearly, this family of imputations between $\mathbf{x}$ and $\mathbf{y}$ contains uncountably infinitely many imputations, as the open line segment is a continuum.

Corollary 3.2. *The image set of the function defined in Lemma 3.1 is uncountable.*

Proof. By CSB, $\mathbb{R} \cong (0,1) \hookrightarrow \{(1-t)\mathbf{x}+t\mathbf{y} \in IMP | t \in (0,1)\} \subseteq IMP \subseteq \mathbb{R}^m \cong \mathbb{R}$. □

The continuum nature of the simplex allows us to "interpolate" a domination relation along the line segment joining an imputation and another imputation it dominates.

Theorem 3.3. *(Interpolation theorem) For $\mathbf{x},\mathbf{y} \in IMP$ and $C \subseteq N$, if $\mathbf{x} \to_C \mathbf{y}$, then $(\forall t \in (0,1))\, \mathbf{x} \to_C (1-t)\mathbf{x}+t\mathbf{y} \to_C \mathbf{y}$.*

Proof. Let $t \in (0,1)$ be arbitrary. We prove $\mathbf{x} \to_C (1-t)\mathbf{x}+t\mathbf{y}$ and $(1-t)\mathbf{x}+t\mathbf{y} \to_C \mathbf{y}$.

For the first domination, as $\mathbf{x} \to_C \mathbf{y}$, we know that $\sum_{k\in C} x_k \leq v(C)$. Further, $(\forall k \in C)\, x_k > y_k$. Let $k \in C$ be arbitrary, then we have $x_k > (1-t)x_k + ty_k \Leftrightarrow tx_k > ty_k \Leftrightarrow x_k > y_k$ (as $t > 0$), which is true. Therefore, $\mathbf{x} \to_C (1-t)\mathbf{x}+t\mathbf{y}$.

For the second domination, as $\mathbf{x} \to_C \mathbf{y}$, we know that $\sum_{k\in C} x_k \leq v(C)$. Further, $(\forall k \in C)\, x_k > y_k$. The second property means $\sum_{k\in C} x_k > \sum_{k\in C} y_k$. Therefore, $\sum_{k\in C} y_k \leq v(C)$. Now consider the quantity $\sum_{k\in C}[(1-t)x_k + ty_k]$. This is equal to $(1-t)\sum_{k\in C} x_k + t\sum_{k\in C} y_k \leq (1-t)v(C) + t\sum_{k\in C} y_k \leq (1-t)v(C) + tv(C) = v(C)$. Therefore, $\sum_{k\in C}[(1-t)x_k + ty_k] \leq v(C)$. Now for $k \in C$, $(1-t)x_k + ty_k > y_k \Leftrightarrow (1-t)x_k > (1-t)y_k$. As $t < 1$, we have $x_k > y_k$, which is true. Therefore, $(1-t)\mathbf{x}+t\mathbf{y} \to_C \mathbf{y}$. □

Theorem 3.3 also has the following consequences for whether the concepts in [6] apply: such AFs are not finitary (Corollary 3.4), not well-founded (Corollary 3.6), not uncontroversial (Corollary 3.7) and not limited controversial (Corollary 3.8).

Corollary 3.4. *If $\langle IMP, \to \rangle$ has a non-empty domination relation, then it is not finitary.*

Proof. If $\to \neq \varnothing$, then there are distinct $\mathbf{x},\mathbf{y} \in IMP$ such that for some non-empty $C \subseteq N$, $\mathbf{x} \to_C \mathbf{y}$. Therefore, $\{(1-t)\mathbf{x}+t\mathbf{y} \in IMP | t \in (0,1)\} \subseteq \mathbf{y}^- \subseteq IMP$, which means $\mathbf{y} \in IMP$ has uncountably infinitely many attackers. The result follows. □

We now generalise Theorem 3.3 to be able to compare two interpolated imputations along the open line segment between them.

Theorem 3.5. *(Double interpolation theorem) For $\mathbf{x},\mathbf{y} \in IMP$ and $C \subseteq N$, if $\mathbf{x} \to_C \mathbf{y}$, then $(\forall s,t \in (0,1))$, if $s < t$, then $\mathbf{x} \to_C (1-s)\mathbf{x}+s\mathbf{y} \to_C (1-t)\mathbf{x}+t\mathbf{y} \to_C \mathbf{y}$.*

Proof. For $\mathbf{z} := (1-s)\mathbf{x}+s\mathbf{y} \to_C \mathbf{y}$, let $u := \frac{t-s}{1-s} \in (0,1)$. Clearly, $(1-u)\mathbf{z}+u\mathbf{y} = (1-t)\mathbf{x}+t\mathbf{y}$, and by Theorem 3.3, $(1-s)\mathbf{x}+s\mathbf{y} \to_C (1-t)\mathbf{x}+t\mathbf{y} \to_C \mathbf{y}$. □

It follows from this that all such continuum AFs are not well-founded.

Corollary 3.6. *For $\langle IMP, \to \rangle$, if $\to \neq \varnothing$, then it is not well-founded.*

Proof. As $\rightarrow \neq \varnothing$, then consider $\mathbf{x} \rightarrow_C \mathbf{y}$. By Theorem 3.5, we have $s,t \in (0,1)$ such that if $s < t$, then $\mathbf{x} \rightarrow_C (1-s)\mathbf{x} + s\mathbf{y} \rightarrow_C (1-t)\mathbf{x} + t\mathbf{y} \rightarrow_C \mathbf{y}$. Define $\mathbf{z}_n := \left(1 - \frac{1}{2^n}\right)\mathbf{x} + \frac{1}{2^n}\mathbf{y}$, for $n \in \mathbb{N}^+$. Clearly, $\mathbf{z}_{n+1} \rightarrow_C \mathbf{z}_n$ by Theorem 3.5. Therefore, the *IMP*-sequence $\{\mathbf{z}_n\}_{n\in\mathbb{N}^+}$ is an infinite backwards attacking chain, thus $\langle IMP, \rightarrow \rangle$ is not well-founded. □

Additionally, Theorem 3.3 shows that there is always a controversial argument.

Corollary 3.7. *For $\langle IMP, \rightarrow \rangle$, if $\rightarrow \neq \varnothing$, then the AF is not uncontroversial.*

Proof. As $\mathbf{x} \rightarrow_C \mathbf{y}$ means $\mathbf{x}$ (indirectly) attacks $\mathbf{y}$. We choose $t = \frac{1}{2} \in (0,1)$ in Theorem 3.3 such that $\mathbf{x} \rightarrow_C \frac{1}{2}(\mathbf{x} + \mathbf{y}) \rightarrow_C \mathbf{y}$, thus $\mathbf{x}$ indirectly defends $\mathbf{y}$. Therefore, $\mathbf{x}$ is controversial w.r.t. $\mathbf{y}$, which means $\langle IMP, \rightarrow \rangle$ is not uncontroversial. □

We show the weaker result of limited controversial is also never true.

Corollary 3.8. *For $\langle IMP, \rightarrow \rangle$, if $\rightarrow \neq \varnothing$, then the AF is not limited controversial.*

Proof. We construct an *IMP*-sequence $\{\mathbf{z}_k\}_{k\in\mathbb{N}}$ such that $(\forall k \in \mathbb{N})\, \mathbf{z}_{k+1}$ is controversial w.r.t. $\mathbf{z}_k$. Consider the infinite backwards attack chain from Corollary 3.6, such that for each $k \in \mathbb{N}$ and $\mathbf{z}_{k+1} \rightarrow_C \mathbf{z}_k$, we apply Theorem 3.3 with $t = \frac{1}{2}$, $\mathbf{x} = \mathbf{z}_{k+1}$ and $\mathbf{y} = \mathbf{z}_k$ to show that $\mathbf{z}_{k+1}$ also defends $\mathbf{z}_k$, and hence $\mathbf{z}_{k+1}$ is controversial w.r.t. $\mathbf{z}_k$, for all $k \in \mathbb{N}$. □

In summary, we have used the property of affine closure in a simplex to interpolate the domination $\mathbf{x} \rightarrow_C \mathbf{y}$ such that every payoff between $\mathbf{x}$ and $\mathbf{y}$ is attacked by $\mathbf{x}$ and attacks $\mathbf{y}$. It follows that $\langle IMP, \rightarrow \rangle$ is not finitary because $\mathbf{y}$ has uncountably infinitely many attackers. Further, $\langle IMP, \rightarrow \rangle$ is not well-founded because one can have an infinite backwards attack sequence from $\mathbf{y}$ with limit $\mathbf{x}$. Also, every intermediate point between $\mathbf{x}$ and $\mathbf{y}$ means that $\mathbf{x}$ is controversial w.r.t. $\mathbf{y}$, and interpolation means $\langle IMP, \rightarrow \rangle$ is also not limited controversial. From the perspective of abstract argumentation, the failure of these properties means we cannot invoke some results of [6, Section 2] to infer further properties of these AFs, e.g. that being uncontroversial means all preferred extensions are stable [6, Theorem 33(2)]. This means continuum AFs like those arising from cooperative games are non-trivial objects to analyse.

4. Conclusions, Future Work and Related Work

In this paper, we have investigated the continuum AFs arising from *m*-player essential transferable-utility cooperative games. In these AFs, the arguments represent the payoff distributions of all *m* players working together , and the attacks represent defection of some of the *m* players where they would each earn strictly more payoff . These AFs are "continuum" as they contain uncountably infinitely many arguments. We have shown that these AFs have several properties that are unlike finite AFs: they are not finitary, not well-founded, not uncontroversial, and not limited controversial. These results are important because they entail that such continuum AFs are challenging to deal with as we cannot simply use the results of [6, Section 2] to infer further properties.

As mentioned in Section 3, future work includes investigating conditions in which these continuum AFs are coherent and relatively grounded, which is challenging as our results show we cannot make use of simplifications such as [6, Theorem 33]. This could potentially contribute to cooperative game theory as articulating the conditions on $\langle N, v \rangle$ for when stable sets exist in $\langle IMP, \rightarrow \rangle$ is non-trivial; this is partly why game theorists moved away from cooperative games in the late 1970s [16]. Future work will investigate what further insights argumentation theory can offer.

As mentioned in Section 2, this paper builds on [6, Section 3.1] and [22]. However, we are not the first to investigate infinite AFs; they were investigated in [6] and furthered in [3] where general existence and uniqueness questions for extensions in infinite AFs are shown in an abstract setting. In contrast, this paper has provided an "authentic" example of infinite AFs that arise from cooperative games. We hope that future work will encourage further exchanges of ideas between argumentation and cooperative games.

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