King’s College London - Taster Lecture

On Search Algorithms of Egg Dropping

David Kohan Marzagão

Department of Informatics

King´s College London

The Problem

The Problem

- There is a building with $f = 100$ floors.
- You have two identical eggs (or balls).
- When thrown through the window of a given floor, an egg can either break or stay intact. **(depends on the height!)**
- Of course, if the eggs break tested from floor n, then they also break if dropped from any floor m such that $m \geq n$.

> **Question** We are interested in the highest floor from which a ball will not break. The goal is to find the minimum number of tests k needed in order to find this floor in the *worst case* scenario.

- Finally, note that there are only two eggs, and if both break, there is no more way to do tests! While an egg doesn't break, it can be used again.

Warming up

First Attempt

- How about the following strategy:

> **Strategy 1** I drop from floor 1, then 2, and so on until it breaks. If breaks from 1, then the answer is 0.

- What is the worst case number of drops using Strategy 1?

1) 1
2) 10
3) 50
4) 99
5) 100

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  4) 99
  5) 100

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First Attempt

- How about the following strategy:

> **Strategy 1** I drop from floor 1, then 2, and so on until it breaks. If breaks from 1, then the answer is 0.

- What is the worst case number of drops using Strategy 1?

1) 1 ✗
2) 10 ✗
3) 50 ✗
4) 99 ✗
5) 100 ✓ **(we need the last drop as it might never break!)**

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However...

- Although the solution above is not the best, it is not useless in our analysis.
- The solution above correspond to the simpler problem in which there is only one egg, not two.
- It is useful now to introduce some notation. **(d stands for 'drops')**

> **Definition** We denote by $d(f, e)$ the solution for the problem (i.e., fewer numbers of drops in the worst case) for f floors and e eggs.

- So we now have a solution for the (general) simpler problem:

> **Proposition** The solution of the minimum number of drops for f floors and $e = 1$ is
> $$d(f, 1) = f \tag{1}$$

- How to prove this is the best? **(assume you can do in less...)**

A digression

- How easy does the problem become if we have as many eggs as we want?
- In other works, what is the value of "$d(100, \infty)$"?

    1) 6
    2) 7
    3) 8
    4) 10
    5) 11

A digression

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- In other works, what is the value of "$d(100, \infty)$"?

  1) 6
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  3) 8
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- **Hint:** Use binary search!

A digression

- How easy does the problem become if we have as many eggs as we want?
- In other works, what is the value of "$d(100, \infty)$"? **(use binary search!)**
  1) 6 ✗
  2) 7 ✓ **(because $\log_2(100) \approx 6.6$)**
  3) 8 ✗
  4) 10 ✗
  5) 11 ✗
- If you want to avoid the use of ∞ you can say that $d(100, 100) = 7$. Or, in general,

Proposition The solution of the minimum number of drops for f floors and $e = f$ eggs is

$$d(f, f) = \lceil \log_2(f) \rceil \quad (2)$$

Combining

- We can combine the two solutions for the simpler problems in order to get bounds for our original challenge.
  - If with one egg we can do in 100 drops, then surely we don't need more for two eggs.
  - On the other hand, we need at least 7.
- Thus:

$$7 \leq d(100, 2) \leq 100$$

Recall that $d(100, 2)$ represents our original problem: we want to find the fewer number of drops needed to determine the threshold floor where $n = 100$ floors, and $e = 2$ eggs.

The Original Problem

Back to the original problem

- Let us look for some new strategies. For example:

> **Strategy #2** Drop from floor $\mathbf{2}$, then $\mathbf{4}$, then $\mathbf{6}$ and so on until it breaks. Then use second egg to check one above of last safe floor.

- What is the worst case scenario of this algorithm?

Back to the original problem

- Let us look for some new strategies. For example:

> **Strategy #2** Drop from floor $\mathbf{2}$, then $\mathbf{4}$, then $\mathbf{6}$ and so on until it breaks. Then use second egg to check one above of last safe floor.

- What is the worst case scenario of this algorithm?
- The worst case scenario of Strategy $\#\mathbf{2}$ is $\mathbf{51}$.
- What next?

A family of strategies

- You must be thinking something like...

> **Strategy #3** This is actually a. family of strategies: drop from floor x, then $2x$, then $3x$ and so on until it breaks. Then use second egg to drop from one above of last safe floor and successively one by one until it breaks.

- Let us explore values of x.

A family of strategies

- You must be thinking something like...

> **Strategy #3** This is actually a. family of strategies: drop from floor x, then $2x$, then $3x$ and so on until it breaks. Then use second egg to drop from one above of last safe floor and successively one by one until it breaks.

- Let us explore values of x.
- You must have came up with a formula that look like this

$$\frac{100}{x} + x - 1$$

- To be precise, you can round this number down (instead of up).
- If you know how to find minima by differentiating functions, now is the time!
- If not, no problem. This is minimised for $x = 10$ and number around it.

What now?

- Can we now be sure that 19 is the number we are looking for?
- How can we **prove** that this works?

What now?

- Can we now be sure that 19 is the number we are looking for?
- How can we **prove** that this works?

> **Remark** Showing that a strategy is the best among a family of strategies does not prove it is the best overall.

- In fact, 19 is **not** the solution.

Flipping the problem

- Try exploring this ‘upside down’ version of the problem.

> **Question** We are interested in the **tallest building** that can be covered with e eggs and d drops. More formally, we want to find $f(e, d)$.

- What is the tallest building we can cover with 2 eggs and 3 drops, this is, $f(2, 3)$?

  1) 3
  2) 4
  3) 5
  4) 6
  5) 7

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- What is the tallest building we can cover with 2 eggs and 3 drops, this is, $f(2, 3)$?

  1) 3 ✗
  2) 4 ✗
  3) 5 ✗
  4) 6 ✓ **(but how?)**
  5) 7 ✗

Flipping the problem

- The solution for the previous problem is $\mathbf{6}$ by doing the following:
    - First we try floor $\mathbf{3}$.
    - If breaks, try from floor $\mathbf{1}$.
    - Otherwise, drop from floor $\mathbf{5}$. **(why not 6?)**

Flipping the problem

- The solution for the previous problem is $\mathbf{6}$ by doing the following:
  - First we try floor $\mathbf{3}$.
  - If breaks, try from floor $\mathbf{1}$.
  - Otherwise, drop from floor $\mathbf{5}$. **(why not 6?)**
  - If breaks, try $\mathbf{4}$.
  - Otherwise, try $\mathbf{6}$.
  - End of available attempts.
- Now consider that number of available drops is $\mathbf{4}$. What is the tallest building that can be covered?
  - Try to think of the highest floors possible for the first drop.

Flipping the problem

- The solution for the previous problem is 6 by doing the following:
  - First we try floor 3.
  - If breaks, try from floor 1.
  - Otherwise, drop from floor 5. **(why not 6?)**
  - If breaks, try 4.
  - Otherwise, try 6.
  - End of available attempts.
- Now consider that number of available drops is 4. What is the tallest building that can be covered?
  - Try to think of the highest floors possible for the first drop.
- The answer is 10 floors. Do we see a pattern?

Lessons learned

- First, studying this flipped problem showed us that there is no reason to have constant intervals from one drop to the next while the first egg doesn't break.
- We found out that by adding a 4th drop, we increased by 4 the number of total floors (from 6 to 10). Is that a coincidence?

Lessons learned

- First, studying this flipped problem showed us that there is no reason to have constant intervals from one drop to the next while the first egg doesn't break.
- We found out that by adding a 4th drop, we increased by $\mathbf{4}$ the number of total floors (from $\mathbf{6}$ to $\mathbf{10}$). Is that a coincidence?
- If we increase the number of drops to $\mathbf{5}$, we should have then $\mathbf{15}$ as the tallest building. It seems we have something like:

$$\mathbf{1, 3, 6, 10, 15 \ldots}$$

Strategy #4 We first try from floor $\boldsymbol{x}$, then $\mathbf{2}\boldsymbol{x} - \mathbf{1}$, then $\mathbf{3}\boldsymbol{x} - \mathbf{2}$ until it first egg breaks. Then, use second egg to drop from one by one from the next above the last higher safe floor.

Uneven Jumps

Strategy #4 We first try from floor x, then $2x - 1$, then $3x - 2$ until it first egg breaks. Then, use second egg to drop from one by one from the next above the last higher safe floor.

- So, what is then the number of tries needed for 100 floors and 2 eggs with strategy $\#4$?

  1) 13
  2) 14
  3) 15
  4) 16
  5) 17

Uneven Jumps

Strategy #4 We first try from floor x, then $2x - 1$, then $3x - 2$ until it first egg breaks. Then, use second egg to drop from one by one from the next above the last higher safe floor.

- So, what is then the number of tries needed for 100 floors and 2 eggs with strategy $\#4$?

  1) 13 ✗
  2) 14 ✓
  3) 15 ✗
  4) 16 ✗
  5) 17 ✗

Uneven Jumps

- Are we done?

> **Conjecture** The solution for our original problem is $\mathbf{14}$. In other words,
>
> $$f(100, 2) = 14$$

Uneven Jumps

- Are we done?

> **Conjecture** The solution for our original problem is 14. In other words,
>
> $$f(100, 2) = 14$$

- **YES**. But... how to prove it?
- **Proof idea:** Assume it is possible to do with less than 14 drops. Then, the first floor we drop from cannot be 14 (why?), it has to be 13 or less. But then, we must be able to cover a building of at least 87 floors in 12 drops. Can we? Why?

(More) General Solutions

- With that in mind, we can now give the general solution based on the formula of the sum of consecutive integers.

> **Theorem** The necessary number of drops to find the threshold with 2 eggs and a building with f floors is
>
> $$d(f, 2) = \min\left\{x \text{ such that } \frac{x^2 + x}{2} \geq f\right\}$$

- Now, the ultimate challenge:

(More) General Solutions

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Theorem The necessary number of drops to find the threshold with 2 eggs and a building with f floors is

$$d(f, 2) = \min \left\{ x \text{ such that } \frac{x^2 + x}{2} \geq f \right\}$$

- Now, the ultimate challenge:

Question How do we generalise this solution 3 eggs? What about e eggs in general?

- Have fun!

Final Comments

- If you want to practice programming, this is a very good problem to explore.
    - **Suggestion:** Try implementing each strategy we say in this class and check the worst case results.
- Today we looked for exact solutions. In some cases of computer science, researches are more interested in the "bigger picture" rather than the exact number involved in the solution.
    - When we study **complexity** of algorithms, to say a procedure needs a number of drops that is *roughly* the square root of the number of floors might be enough.
- The exact solution for the generalised problem is not that easy. Try exploring easier cases first. **(hint: pascal triangle is somehow involved)**

The End

(you can find a copy of the slides at `https://kohan.uk/fun-stuff/`)